On the non-existence of 't Hooft-Polyakov magnetic monopoles in non-semisimple electroweak gauge theories

1984 J. Phys. A: Math. Gen. 17 3287

(http://iopscience.iop.org/0305-4470/17/17/006)

J. Phys. A: Math. Gen. 17 (1984) 3287-3298. Printed in Great Britain

On the non-existence of ’t Hooft–Polyakov magnetic monopoles in non-semisimple electroweak gauge theories

B T McInnes

Department of Mathematics, National University of Singapore, Kent Ridge 0511, Republic of Singapore

Received 21 November 1983, in final form 21 May 1984

Abstract. The non-existence of stringless monopole solutions in the Weinberg–Salam theory (except in the case in which the electromagnetic U(1) is contained in the SU(2) factor of the gauge grou p) is usually ascribed to a combination of circumstances arising from the topology of the gauge group, from the underlying Lie algebra structure, and from the nature of the Higgs mechanism. Here we analyse this result from a geometric point of view, and show that no electroweak theory based on a compact connected non-semisimple group can admit stringless monopoles, unless the charge operator lies in the semisimple part of the Lie algebra. The analysis is independent of the mechanism used to break the symmetry.

1. Introduction

It is well known that the classical Yang–Mills–Higgs field equations admit non-singular finite energy solutions corresponding to magnetic monopoles (’t Hooft 1974, Polyakov 1974; see Goddard and Olive 1978, Actor 1979, Craigie *et al* 1982 for reviews). The existence of such ‘particle-like’ solutions has naturally attracted great interest, particularly since it seems probable that grand unified gauge theories admit solutions with the same properties. Yet it has long been known that neither the Weinberg–Salam unified electroweak theory, nor quantum chromodynamics, admits such solutions at the classical level. (See Huang (1982) for a clear discussion of this point.) In view of the fact that these two theories are at present the only gauge theories with substantial experimental support, this remark is clearly of some significance. The non-existence of ‘stringless’ monopoles in the standard strong–electroweak SU(3) × SU(2) × U(1) model can be regarded as an experimental prediction—a prediction which sharply distinguishes this model from any grand unification scheme. As the experimental situation regarding the abundance of monopoles becomes clearer, this prediction may prove to be decisive.

In view of these remarks, it is important that the basis of the SU(3) × SU(2) × U(1) monopole non-existence theorem be clearly understood. In the case of quantum chromodynamics, the argument is straightforward: the non-existence of static stringless monopoles follows directly from the energy finiteness condition. In fact, Deser (1976) has shown quite generally that the pure gauge field equations (that is, with no sources) for a connected compact semisimple gauge group admit no static finite energy solutions in Minkowski space.

The situation in the Weinberg–Salam case is rather less clearcut. Deser’s result is not applicable here; instead, it is argued that the monopole non-existence result arises because of the inability of the Higgs fields to realise their boundary conditions at spatial infinity in a topologically non-trivial way. This property of the Higgs fields may in its turn be related to the topological structure of the gauge group (see Goddard and Olive 1978). Without going into the details, we merely observe that this argument depends on the details both of the Higgs mechanism and of the global structure of the gauge group.

The dependence of the monopole non-existence result on the workings of the Higgs mechanism is noteworthy, since this mechanism is widely regarded as the least satisfactory aspect of the Weinberg–Salam model. Although no entirely satisfactory substitute has yet been proposed, it seems quite conceivable that the Higgs mechanism may ultimately be replaced by some other symmetry-breaking technique. It is natural to ask whether such a replacement would affect the non-existence theorem. In a similar vein, let us remark that the global structure of $SU(2)\times U(1)$ is rather remote from most phenomenological applications of the Weinberg–Salam theory, it being the Lie *algebra* which determines, for example, the spectrum of intermediate vector bosons. The dependence of the monopole non-existence theorem on the global structure of $SU(2)\times U(1)$ thus leads us to ask whether the theorem can be circumvented by replacing $SU(2)\times U(1)$ by a locally isomorphic group with a different global structure.

Although these remarks are of course highly speculative, they do serve to underline the importance of establishing a precise basis for the assertion that stringless monopoles cannot arise from the electroweak fields alone. It should perhaps be remarked that if it were indeed possible in some way to circumvent the non-existence theorem, the corresponding monopoles could well be relatively light and thus ultimately accessible to direct experimentation. We therefore ask: what are the *minimal* assumptions necessary to exclude 't Hooft–Polyakov-type monopoles from gauge theory based on the Weinberg–Salam algebra?

From a mathematical point of view, gauge theory finds its most natural formulation in terms of the geometry and topology of fibre bundles (see, for example, Trautman 1970, Drechsler and Mayer 1977, Daniel and Viallet 1980, Bleecker 1981). The geometric description is particularly appropriate in the case of magnetic monopoles (Wu and Yang 1975). We therefore expect that it may be of great interest to analyse the Weinberg–Salam monopole non-existence theorem from this point of view. The objective of the present work is to carry out such an analysis at the greatest possible level of generality, with the intention of obtaining a unified view of the non-existence of stringless monopoles in certain gauge theories. Apart from its intrinsic interest, the fibre bundle approach has the virtue that it now proves to be possible to derive the relevant results with far fewer initial assumptions than are required in the usual analysis. In particular, we shall find that the non-existence of monopoles in the Weinberg–Salam theory can be deduced without any information whatever regarding the behaviour (or indeed the existence) of the Higgs fields. Further, it will be seen that the global structure of the gauge group plays only a minor role in this discussion. The key feature of $SU(2)\times U(1)$ from this point of view is its Lie *algebra*. These conclusions virtually rule out any possibility of modifying the Weinberg–Salam theory to accommodate stringless monopoles (except in the unphysical case in which the electromagnetic $U(1)$ is contained in the $SU(2)$ factor), other than by considering grand unified theories.

2. The geometric characterisation of stringless monopoles

The geometric structures underlying the Dirac and 't Hooft–Polyakov monopoles have been discussed in the literature (for example, Trautman 1979, Goddard and Olive

1978, Quiros *et al* 1982) and need only be examined here in order to stress certain salient features.

It is well known that the field of a static magnetic monopole, which we regard here as a U(1) gauge field, cannot be represented by a single vector potential which is well defined everywhere in the region exterior to the monopole. This is a result of the fact that a monopole field arises from a connection in a non-trivial principal fibre bundle which does not admit a global section. Any attempt to use a single potential gives rise to an artificial singularity, the so-called 'Dirac string'.

The various non-trivial principal bundles over a given base manifold are classified according to the general theory of characteristic classes (see, for example, Kobayashi and Nomizu 1969), which in the case of U(1) reduce to a special case of the Chern classes. The integral of the first Chern class for a monopole field over a 2-sphere enclosing the monopole is related to the latter's magnetic charge. From this statement it is clear that in order for a magnetic monopole to exist, *it is essential that the electromagnetic U*(1) *buncle should admit non-zero primary characteristic classes.* This alone is sufficient for our purposes; we need not go further into the details of the topological interpretation of magnetic charge (see Quiros *et al* 1982).

Now let us turn to the 't Hooft–Polyakov monopole, which corresponds to a solution of the combined Yang–Mills–Higgs field equations. If these equations are interpreted physically as the basis of a unified electroweak theory, with SU(2) breaking down to U(1) (see Georgi and Glashow 1972), then the electromagnetic part of the 't Hooft–Polyakov field does indeed correspond to the field of a magnetically charged particle. Yet the full solution is free of all singularities. As Trautman (1979) has pointed out, this has the following simple geometric interpretation. The absence of 'strings' at the SU(2) level means that the 't Hooft–Polyakov field arises from a connection in a *trivial* principal SU(2) bundle. Now the breakdown of SU(2) to U(1) corresponds to the reduction (Kobayashi and Nomizu (1963) of this SU(2) bundle to the electromagnetic U(1) bundle. The key point now is that the triviality of a principal bundle does not necessarily impose triviality on all of its sub-bundles. Thus we see how a magnetic monopole can arise, through symmetry breaking, from connections on a trivial principal bundle. This is, in fact, the characteristic topological structure underlying the 't Hooft–Polyakov monopole: a *trivial* SU(2) principal bundle which admits a *non-trivial* U(1) sub-bundle.

For an arbitrary gauge theory based on a Lie group G which breaks down to a closed subgroup H, we shall define a generalised stringless monopole of type (G, H) to be a localised finite energy field configuration corresponding to a connection in a trivial principal bundle (P, M, G) which admits a non-trivial sub-bundle (Q, M, H). Of course, the mere existence of the basic topological structure is not in itself sufficient to guarantee the existence of monopoles: in particular, the energy finiteness condition cannot always be satisfied. In the case of the 't Hooft–Polyakov monopole, the Higgs fields play an essential role in rendering the energy finite; more generally, it is always necessary to introduce some kind of matter field in order to obtain finite energy. Thus it is never possible to establish the existence of monopoles within a given theory on the basis of topological or geometrical considerations alone. However, this comment is not valid when we attempt to show that some gauge theory does *not* admit monopoles. It may happen, for example, that for some particular choice of G and H, no trivial G bundle admits a non-trivial H subbundle. In such a case, monopoles can be ruled out *a priori*, independently of the existence of Higgs or other fields. As will now be shown, this point of view permits the construction of monopole non-existence results on the basis of very little information.

3. Non-existence of stringless gravitational monopoles

In this section we discuss the non-existence of generalised stringless monopoles in the gravitational case. This may be regarded as a particularly simple illustration of the general approach being developed in the present work, although the result is of interest in its own right.

One of the most attractive features of the bundle-theoretic approach to gauge theories is the fact that such a formulation automatically unifies the mathematical descriptions of the gravitational and non-gravitational fields. (Any space-time linear connection can be regarded as the 'pull-back' of a connection on the bundle of linear frames over space–time.) Unfortunately, there appear to be serious difficulties, particularly at the dynamical level, in pursuing this analogy to its logical conclusion, and so the question of whether gravitation should be considered as a gauge field remains open. However, it is clear that any 'gauge theory of gravitation' should incorporate the observation that gravity is directly related to the structure of space–time, whereas such does not appear to be the case for the other interactions. This aspect of the gravitational field can be accommodated by restricting the gravitational gauge bundle to be 'soldered' to the base manifold (Trautman 1970). That is, we take the principal bundle to be one of the various frame bundles over space–time. (See Kobayashi 1972, Trautman 1976; henceforth we treat only the case of the linear frame bundle, since theories based on the other frame bundles (affine, projective, etc) can be treated similarly.)

The role of the metric in such a gauge theory is somewhat analogous (Trautman 1979) to that of a Higgs field in a standard gauge theory, in the sense that the existence of a space–time metric reduces the bundle of linear frames to a sub-bundle of pseudo-orthonormal frames. That is, the metric 'breaks' $GL(4, \mathbb{R})$ (the group of real 4×4 invertible matrices) down to the Lorentz group. According to the discussion of the preceding section, then, a gravitational analogue of a stringless monopole would be constructed around the following basic topological structure: a trivial frame bundle admitting a non-trivial Lorentz sub-bundle. No such structure is possible, however. The triviality of the frame bundle is equivalent to the existence of a smooth global moving frame; the metric can then be used to construct a global pseudo-orthonormal moving frame. This in its turn gives a global cross section of the Lorentz sub-bundle, which must therefore be trivial. Thus we conclude at once that gravitational analogues of 't Hooft–Polyakov monopoles do not exist. (See the recent work of Ross (1983) on the non-existence of Dirac gravitational monopoles.)

In the preceding section it was pointed out that, in general, it is possible for a trivial principal bundle to admit a non-trivial sub-bundle. In the present case, however, this does not occur: the triviality of the frame bundle imposes a similar condition on any Lorentz sub-bundle. We shall say in such a case that the sub-bundle 'inherits' triviality from the original bundle. In order that generalised stringless monopoles of type (G, H) should exist, it is clearly necessary (but by no means sufficient) that H sub-bundles of trivial G bundles should not always inherit triviality. Our immediate objective in the present work is to show that 'inheritance of triviality' is responsible for the non-existence of stringless monopoles in the Weinberg-Salam theory: that is, similar mathematical conditions underlie the gravitational and electroweak monopole non-existence results. The concept of inheritance can be analysed with the aid of the general theory of characteristic classes, to which we now turn.

4. Characteristic classes

Let G be an arbitrary Lie group with Lie algebra $\mathscr{G}$, and let $I^k(\mathrm{G})$ denote the vector space of symmetric, Ad-invariant, multilinear mappings $f: \mathscr{G}^k \to \mathbb{R}$. (If G is Abelian, then $I^k(\mathrm{G})$ is just the space of symmetric mappings $f: \mathscr{G}^k \to \mathbb{R}$.) Let (P, M, G) be any principal G bundle, with projection π, over a base manifold M. If Γ is any connection on P, let Ω be the curvature form. Then for each $f \in I^k(\mathrm{G})$, we define a real-valued $2k$-form $f(\Omega)$ on P by (Kobayashi and Nomizu 1969)

$$f(\Omega)(p)(X_1 \cdots X_{2k}) = \frac{1}{(2k)!} \sum_\sigma \mathrm{sgn}(\sigma) f(\Omega(p)(X_{\sigma(1)}, X_{\sigma(2)}), \ldots \Omega(p)(X_{\sigma(2k-1)}, X_{\sigma(2k)}))$$

where $p \in P$, $X_1 \ldots X_{2k}$ are tangent vectors at P, and the summation is over all permutations σ of $\{1 \ldots 2k\}$. It is possible to show that for each $f(\Omega)$ so defined, there exists a unique $2k$-form $\bar{f}(\Omega)$on M such that

$$\pi^*(\bar{f}(\Omega)) = f(\Omega).$$

Like $f(\Omega)$, $\bar{f}(\Omega)$ depends on the choice of Γ. However, the Bianchi identity implies that $\bar{f}(\Omega)$ is closed, and an important result due to Weil states that the corresponding de Rham cohomology class $[\bar{f}(\Omega)]$ is actually independent of the choice of Γ. This class is therefore determined purely by f and by the structure of (P, M, G), and is called the characteristic class corresponding to f. In particular, we shall refer to the $k = 1$ classes as the primary characteristic classes.

Since every trivial principal bundle admits a canonical connection with $\Omega = 0$, Weil's result implies that all characteristic classes vanish for a trivial bundle. In view of the discussions of the preceding sections, it is clearly a prerequisite for the existence of stringless monopoles that the vanishing of all characteristic classes of a principal bundle should not necessarily annihilate the characteristic classes of its sub-bundles. Let us investigate this point.

Let H be a closed Lie subgroup of G, and let (Q, M, H) be a reduced sub-bundle of (P, M, G). We denote the embedding of Q in P by $b: Q \to P$. For each k, there is a natural restriction map $\rho_k: I^k(\mathrm{G}) \to I^k(\mathrm{H})$, since any Ad(G)-invariant multilinear mapping is automatically Ad(H) invariant upon being restricted to the Lie algebra $\mathscr{H}$ of H.

Now assume that (P, M, G) is trivial. Let $f_0 \in I^k(\mathrm{H})$ be such that there exists $f \in I^k(\mathrm{G})$ with $\rho_k: f \to f_0$; that is, take $f_0 \in \rho_k I^k(\mathrm{G})$. Let Γ_0 be an arbitrary connection on Q, with curvature form Ω_0. Then there exists a unique connection Γ on P with curvature from Ω such that $b^*\Omega = \Omega_0$ (Kobayashi and Nomizu 1963). Let $f_0(\Omega_0)$ be a real-valued $2k$-form on Q defined as above, let $\bar{f}_0(\Omega_0)$ be the corresponding form on M, and let $X_1 \ldots X_{2k}$ be any $2k$ tangent vectors at $x \in M$. Then if $q \in Q$ satisfies $\pi(q) = x$, and if $\bar{X}_1 \ldots \bar{X}_{2k}$ are tangent vectors at q which satisfy $\pi_* \bar{X}_1 = X_1, \ldots, \pi_* \bar{X}_{2k} = X_{2k}$, we have

$$\begin{aligned}\bar{f}_0(\Omega_0)(x)(X_1 \ldots X_{2k}) &= f_0(\Omega_0)(q)(\bar{X}_1 \ldots \bar{X}_{2k}) \\ &= \frac{1}{(2k)!} \sum_\sigma \mathrm{sgn}(\sigma) f_0(\Omega_0(q)(\bar{X}_{\sigma(1)}, \bar{X}_{\sigma(2)}, \ldots)\end{aligned}$$

$$= \frac{1}{(2k)!} \sum_{\sigma} \mathrm{sgn}(\sigma) f(\Omega(b(q))(b_* \bar{X}_{\sigma(1)}, b_* \bar{X}_{\sigma(2)}), \ldots)$$

$$= f(\Omega)(b(q))(b_* \bar{X}_1 \ldots b_* \bar{X}_{2k})$$

$$= \bar{f}(\Omega)(x)(X_1 \ldots X_{2k}).$$

Thus $\bar{f}_0(\Omega_0) = \bar{f}(\Omega)$, and the respective cohomology classes coincide. The triviality of (P, M, G) therefore directly implies $[\bar{f}_0(\Omega_0)] = 0$. Extending the terminology of the preceding section, we shall say that the characteristic class $[\bar{f}_0(\Omega_0)]$ 'inherits triviality' from (P, M, G). Clearly, the characteristic classes of (Q, M, H) which inherit triviality are precisely those which correspond to the elements of $\rho_k I^k(\mathrm{G})$.

The above discussion serves to clarify the conditions which must be obtained if stringless monopole solutions are to exist within a given gauge theory. Suppose, for example that $\rho_k I^k(\mathrm{G}) = I^k(\mathrm{H})$ for all k. Then the stringless monopoles of type (G, H) cannot exist, since in such a case all characteristic classes of (Q, M, H) inherit triviality. In fact, however, this condition is not usually satisfied, since the Ad(H) invariance of a symmetric multilinear mapping is not sufficient to guarantee that it will be Ad(G) invariant. The characteristic classes corresponding to elements of $I^k(\mathrm{H})$ which do not lie in $\rho_k I^k(\mathrm{G})$ are unaffected by the triviality of (P, M, G), and so it becomes possible for (Q, M, H) to be non-trivial. The initial step in determining whether (G, H) monopoles can exist thus involves a study of the spaces $\rho_k I^k(\mathrm{G})$. We shall carry out this analysis for gauge theories of the electroweak interaction.

Before leaving the general theory, we make the following remarks. Firstly, consider the case in which G is Abelian. Then, as was pointed out earlier, $I^k(\mathrm{G})$ and $I^k(\mathrm{H})$ are simply the spaces of symmetric k-linear real-valued mappings on $\mathscr{G}^k$ and $\mathscr{H}^k$ respectively. It is easily seen that each ρ_k is surjective in this case, that is, $\rho_k I(\mathrm{G}) = I^k(\mathrm{H})$ for every k. Hence, stringless monopoles cannot exist in any Abelian gauge theory—they are a strictly 'non-Abelian' phenomenon.

Secondly, we wish to emphasise the fact that the theory of characteristic classes has enabled us to translate a topological question (concerning the existence of non-trivial sub-bundles of trivial principal bundles) into a question concerning $I^k(\mathrm{G})$ and $I^k(\mathrm{H})$. But the structures of these spaces are determined purely by those of G and H. The entire duscission is altogether independent of the behaviour of the Higgs fields. (Indeed, Weil's theorem implies that it is independent of *all* fields.)

5. The role of inheritance in electroweak theories

We now specialise the above discussion in order to consider stringless monopoles in electroweak theories. In this case, we have a Lie group G (not necessarily semisimple) which breaks down to the electromagnetic U(1) gauge group. The latter is generated by some fixed linear combination of elements drawn from a Cartan subalgebra (denoted by $\mathscr{C}$) of the Lie algebra $\mathscr{G}$. Thus, U(1) is a subgroup of a maximal torus (denoted by T) of G.

As usual, we represent the G gauge fields as 'pull-backs' of connections in a principal bundle (P, M, G). The breakdown of G to U(1) means that (P, M, G) admits a reduced sub-bundle $(E, M, \mathrm{U}(1))$; this is equivalent (Kobayashi and Nomizu 1963) to the existence of a global section of the associated bundle of (P, M, G) with standard fibre

G/U(1). The existence of this section implies the existence of a global section of the associated bundle with standard fibre G/T, so that we may assume that (P, M, G) also admits a reduced sub-bundle (Q, M, T), which we shall call the 'maximal toral sub-bundle'. The electromagnetic bundle is then a sub-bundle of (Q, M, T). It is to be emphasised that we are not assuming here that the symmetry breaking proceeds by stages $\mathrm{G} \to \mathrm{T} \to \mathrm{U}(1)$; we merely propose to make use of the mathematical observation that (Q, M, T) can in fact be constructed.

Our objective is to examine the effect on $(E, M, \mathrm{U}(1))$ of assuming that (P, M, G) is trivial. We do this by first considering the inheritance problem for (P, M, G) and (Q, M, T), and subsequently examining the effects on $(E, M, \mathrm{U}(1))$, regarded as a sub-bundle of (Q, M, T). Since both T and U(1) are Abelian, this latter analysis is relatively straightforward. The central problem is the determination of the set of characteristic classes of (Q, M, T) which inherit triviality from (P, M, G). This can be carried out with the aid of standard theorems.

Given a Cartan subalgebra $\mathscr{C}$ of the Lie algebra $\mathscr{G}$, let N denote its normaliser in the Lie group G. That is

$$\mathrm{N} = \{g \in \mathrm{G} \text{ such that } \mathrm{Ad}(g)\mathscr{C} \subseteq \mathscr{C}\}.$$

Let T be the maximal torus corresponding to $\mathscr{C}$; then T is a normal subgroup of N. For reasons to be discussed later, we refer to N/T as the *generalised Weyl group* of $\mathscr{C}$. If $g \in \mathrm{N}$, $A \in \mathscr{C}$, and $t \in \mathrm{T}$, then $\mathrm{Ad}(gt)A = \mathrm{Ad}(g)A$ since T is Abelian. Thus $\mathrm{Ad}(w)A$ is well defined for any $w \in \mathrm{W} = \mathrm{N}/\mathrm{T}$, and the generalised Weyl group can be regarded as a group of linear transformations of $\mathscr{C}$.

For each k, let $I^k_{\mathrm{w}}(\mathrm{T})$ denote the subspace of $I^k(\mathrm{T})$ which consists of Ad(W)-invariant elements. The following theorem is fundamental (Kobayashi and Nomizu 1969, p 299).

Theorem 1. Let G be a compact Lie group with Lie algebra $\mathscr{G}$, and let $\mathscr{C}$ be a Cartan subalgebra of $\mathscr{G}$ with corresponding maximal torus T. If $\mathscr{G} = \mathrm{Ad}(\mathrm{G})\mathscr{C}$, then each restriction mapping $\rho_k : I^k(\mathrm{G}) \to I^k(\mathrm{T})$ maps $I^k(\mathrm{G})$ isomorphically onto $I^k_{\mathrm{w}}(\mathrm{T})$.

This theorem gives a precise characterisation of $\rho_k I^k(\mathrm{G})$ as the Weyl-invariant subspace of $I^k(\mathrm{T})$. Combining theorem 1 with our earlier discussions, we obtain a theorem specifying the characteristic classes of (Q, M, T) which inherit triviality.

Theorem 2. Let (P, M, G) be a trivial principal bundle with compact structural group G, and let the Lie algebra $\mathscr{G}$ satisfy $\mathscr{G} = \mathrm{Ad}(\mathrm{G})\mathscr{C}$, where $\mathscr{C}$ is a Cartan subalgebra. If T is the corresponding maximal torus, then the characteristic classes of any maximal toral sub-bundle which inherit triviality are precisely those generated by $I^k_{\mathrm{w}}(\mathrm{T})$, for all k.

We now turn to the second part of the analysis mentioned above, that is, to the relationship between the maximal toral sub-bundle (Q, M, T) and the electromagnetic bundle $(E, M, \mathrm{U}(1))$. The Lie algebra of U(1) is a one-dimensional subspace of $\mathscr{C}$; we take a basis $\{q\}$, where q is the charge operator. Henceforth we shall be mainly interested in the case $k = 1$, that is, in the primary characteristic classes, which are closely related to the magnetic charge of a monopole. Assume that there exists an element $f \in I^1_{\mathrm{w}}(T)$ with $f(q) \neq 0$. Let $f_0 \in I^1(\mathrm{U}(1))$ and consider

$$f' = f_0(q) f / f(q).$$

Since $f_0(q)$ and $f(q)$ are just real numbers, f' is an element of $I^1_w(\mathrm{T})$ which, when restricted to the Lie algebra of U(1), coincides with f_0. Since T and U(1) are Abelian, it follows that if the characteristic class generated by f' is zero, then every primary characteristic class of $(E, M, \mathrm{U}(1))$ will vanish. But theorem 2 states that the characteristic class (of (Q, M, T)) generated by f' will vanish if (P, M, G) is trivial, provided that G satisfies certain conditions. These remarks complete our general study of the consequences of taking (P, M, G) to be trivial. The discussion may be summarised as follows.

Theorem 3. Let (P, M, G) be a trivial principal bundle with compact structural group G, let the Lie algebra of G satisfy $\mathscr{G} = \mathrm{Ad}(\mathrm{G})\mathscr{C}$ (where $\mathscr{C}$ is a Cartan subalgebra corresponding to a maximal torus T), and let U(1) be generated by q, a fixed linear combination of elements of $\mathscr{C}$. Then if $I^1_w(\mathrm{T})$ contains any element which does not annihilate q, the primary characteristic classes of any sub-bundle $(E, M, \mathrm{U}(1))$ inherit triviality from (P, M, G).

In physical language, this means that if a Lie group G of the type described is taken as the gauge group of a unified electroweak gauge theory, and if $I^1_w(\mathrm{T})$ contains any element which does not annihilate the charge operator, then the gauge theory in question does not admit stringless magnetic monopoles of the 't Hooft–Polyakov type. The structure of the (generalised) Weyl group is thus seen to play a central role in the question of whether a particular gauge theory gives rise to such monopoles.

The remainder of this work is devoted to an examination of the consequences of theorem 3. First, we show in detail the way in which gauge theories based on semisimple G are able to avoid the inheritance problem, and second, we use theorem 3 to show that inheritance is directly responsible for the non-existence of stringless monopoles in electroweak theories of the Weinberg–Salam type.

6. Semisimple electroweak gauge groups

Since stringless monopoles are known to exist in the SU(2) electroweak theory, it is clear that one or more of the hypotheses of theorem 3 must be violated here. We now examine this, without giving the proofs in full detail.

Let G be a connected semisimple Lie group with Lie algebra $\mathscr{G}$, and let $\mathscr{C}$ be a Cartan subalgebra of $\mathscr{G}$. We wish to show that $\mathscr{G} = \mathrm{Ad}(\mathrm{G})\mathscr{C}$. (See Varadarajan (1974) for the background.) The fact that G is connected allows us to reduce this equation to the following algebraic form. Let $\mathscr{G} = \mathscr{C} \oplus \mathscr{A}$ be a direct sum decompositon. Then we must show that $\mathscr{A} = [\mathscr{A}, \mathscr{C}]$. Now $\mathscr{A}$ has a root-space decomposition, so that any element of $\mathscr{A}$ has the form $\Sigma\, C_\lambda X_\lambda$, where the sum is taken over the (finite) set of roots Δ, and $[\mathrm{H}, X_\lambda] = \lambda(\mathrm{H}) X_\lambda$ where λ is a root, and $\mathrm{H} \in \mathscr{C}$. Let $\{H_\lambda\}$ be the natural basis of $\mathscr{C}$ generated by Δ. Since $\{H_\lambda\}$ is finite, it is always possible to find $\mathrm{H} \in \mathscr{C}$ which is orthogonal (with respect to the Cartan–Killing form) to none of the H_λ. Thus $\lambda(\mathrm{H}) \neq 0$ for every $\lambda \in \Delta$. Given any $\Sigma\, C_\lambda X_\lambda \in \mathscr{A}$, we can construct $\Sigma\, (C_\lambda X_\lambda / \lambda(\mathrm{H})) \in \mathscr{A}$. Then

$$\left(\sum (C_\lambda X_\lambda / \lambda(\mathrm{H})), \mathrm{H}\right) = \sum C_\lambda X_\lambda$$

which means that $\mathscr{A} \subseteq [\mathscr{A}, \mathscr{C}]$. But it is obvious that $[\mathscr{A}, \mathscr{C}] \subseteq \mathscr{A}$, and so we obtain the desired result. Hence $\mathscr{G} = \mathrm{Ad}(\mathrm{G})\mathscr{C}$ for every connected semisimple G.

In order to proceed, we require some information concerning the generalised Weyl group W. It is a remarkable fact that, in the case of a connected compact semisimple group G, W can in principle be determined by *purely algebraic* methods (that is, without further information about the global topology of G). We now discuss this important point.

Let $\{H_\lambda\}$ be the basis of $\mathscr{C}$ introduced above. For each λ, let S_λ be the reflection in the hyperplane orthogonal to H_λ; that is, for any $\mathrm{H} \in \mathscr{C}$,

$$S_\lambda : \mathrm{H} \to \mathrm{H} - (2B(\mathrm{H}, H_\lambda)H_\lambda / B(H_\lambda, H_\lambda))$$

where B is the Cartan–Killing form. Then the group generated by all such S_λ is called the Weyl group. The Weyl group describes the symmetries of the root system Δ, and is thus determined in a purely algebraic way. The following theorem is standard (Varadarajan 1974, p 356).

Theorem 4. Let G be a connected compact semisimple Lie group with Lie algebra $\mathscr{G}$, let $\mathscr{C}$ be a Cartan subalgebra corresponding to a maximal torus T, and let N be the normaliser of $\mathscr{C}$ in G. Then the Weyl group of the relevant root system is isomorphic to N/T.

In the semisimple case, then, the generalised Weyl group W = N/T coincides with the Weyl group as usually defined. Therefore, the space $I^1_w(\mathrm{T})$ may in this case be described as the space of linear mappings $f: \mathscr{C} \to \mathbb{R}$ which are invariant under every Weyl reflection. Clearly this implies $f(H_\lambda) = 0$ for every $\lambda \in \Delta$, and so (since $\{H_\lambda\}$ is a basis) $f = 0$. Thus $I^1_w(\mathrm{T})$ is zero for every compact connected semisimple G: it contains no elements which fail to annihilate the charge operator. From theorem 3, we see that *it is this fact which permits the construction of the topological structure underlying 't Hooft–Polyakov monopoles.*

A number of comments may now be made in connection with this discussion. Firstly, our proof does not of course imply that stringless monopoles can be constructed in every electroweak theory based on a compact connected semisimple gauge group. There are several other physical and mathematical conditions which must also be satisfied if such a construction is to be possible.

A second noteworthy point concerns the type of information on which this demonstration is based. The only strictly global condition imposed on G has been connectedness, since compactness is essentially an algebraic condition for a connected semisimple group. (A well known theorem of Weyl states that such a group is compact if and only if its Cartan–Killing form is negative definite.) The Lie algebra clearly plays the dominant role here; detailed topological information is not required.

Finally, we wish to emphasise the significance of the generalised Weyl group, which here coincides with the Weyl group. The Weyl group has in fact arisen in previous discussions of magnetic monopoles (Goddard *et al* 1977; see also the contributions of Olive and Cho in Craigie *et al* 1982), in connection with the concepts of magnetic weights and duality conjectures. In that application, the Weyl group appears as a kind of gauge ambiguity in the weights. Here, however, the Weyl group plays a much more central role: the 't Hooft–Polyakov-type monopoles may in fact be said to 'owe their existence' to the Weyl symmetry. It should be stressed that the type of monopole being discussed in this paper (with U(1) as the unbroken group) is quite different to that considered by Goddard *et al* (1977); we are merely suggesting that the appearance of the Weyl group in both discussions may be of significance.

7. Non-semisimple electroweak gauge group

We now consider the more realistic case in which G is not semisimple. This covers the Weinberg–Salam theory as well as its most satisfactory competitors (such as the 'ambidextrous' theory of de Rujula *et al* (1977), based on $SU(2)\times SU(2)\times U(1)$).

It is possible to show (Hochschild 1965) that the Lie algebra $\mathscr{G}$ of any compact connected Lie group may be decomposed as $\mathscr{G}=\mathscr{K}\oplus\mathscr{G}_0$, where each element of $\mathscr{K}$ commutes with all elements of $\mathscr{G}$, and where $\mathscr{G}_0$ is a semisimple compact subalgebra: that is, $\mathscr{G}_0$ may be regarded as the Lie algebra of a connected semisimple Lie subgroup of G, which we denote G_0. Let $Y+Z\in\mathscr{G}$, where $Y\in\mathscr{K}$, $Z\in\mathscr{G}_0$. Then by our earlier discussions, there exist $g\in G_0$, $X\in\mathscr{C}_0$ (a Cartan subalgebra of $\mathscr{G}_0$), such that $Z=\mathrm{Ad}(g)X$. But $\mathrm{Ad}(g)Y=Y$, so that $Y+Z=\mathrm{Ad}(g)(Y+X)$. The Cartan subalgebra of $\mathscr{G}$ corresponding to $\mathscr{C}_0$ is $\mathscr{C}=\mathscr{K}\oplus\mathscr{C}_0$; hence we have shown $\mathscr{G}=\mathrm{Ad}(G)\mathscr{C}$ for any compact connected group.

We now consider the generalised Weyl group W. In the non-semisimple case, theorem 4 does not apply; W can no longer be identified with the full reflection group of some basis of $\mathscr{C}$. Rather than attempt to generalise theorem 4, we shall proceed as follows. Writing $\mathscr{C}=\mathscr{K}\oplus\mathscr{C}_0$ as above, let f be any linear map $f:\mathscr{C}\to\mathbb{R}$ which annihilates $\mathscr{C}_0$. Since G is connected, we have $\mathrm{Ad}(G)\mathscr{G}_0\subseteq\mathscr{G}_0$; thus for any $Y+Z\in\mathscr{C}$ and $w\in W$, $\mathrm{Ad}(w)Z\in\mathscr{C}_0$. But $\mathrm{Ad}(w)Y=Y$; therefore $f(\mathrm{Ad}(w)(Y+Z))=f(Y+\mathrm{Ad}(w)Z)=f(Y)=f(Y+Z)$, for any $w\in W$, $Y\in\mathscr{K}$, $Z\in\mathscr{C}_0$. Thus any $f\in I^1(T)$ (where T is the maximal torus corresponding to $\mathscr{C}$) which annihilates $\mathscr{C}_0$ is Ad(W) invariant.

Let Y be any non-zero element of $\mathscr{K}$, and let f be a linear map $f:\mathscr{K}\to\mathbb{R}$ such that $f(Y)\neq 0$; it is obvious that such an f always exists. Extend f to $f:\mathscr{C}\to\mathbb{R}$ by defining $f(\mathscr{C}_0)=0$. Then f is a linear Weyl-invariant map on $\mathscr{C}$ which does not annihilate Y. It is clear from this discussion that if the charge operator q has a non-zero $\mathscr{K}$ component, then there always exists an element of $I^1_w(T)$ which does not annihilate q. From theorem 3, we now obtain theorem 5.

Theorem 5. Let (P, M, G) be a trivial principal bundle with compact connected non-semisimple structural group G, and let q be a linear combination of elements drawn from a Cartan subalgebra of $\mathscr{G}$, the Lie algebra of G. Let U(1) be generated by q, and set $\mathscr{G}=\mathscr{K}\oplus\mathscr{G}_0$, where $\mathscr{G}_0$ is semisimple and each element of $\mathscr{K}$ commutes with $\mathscr{G}$. Then if q has a non-zero $\mathscr{K}$ component, the primary characteristic classes of any sub-bundle $(E, M, U(1))$ inherit triviality from (P, M, G).

In physical language, the $\mathscr{K}$ component of $\mathscr{G}$ corresponds to the weak hypercharge operator. The quantum numbers of weakly interacting fields are assigned in accordance with a fixed charge formula (such as, in the usual Weinberg–Salam case, $q=Y+T_3$, where Y is hypercharge and T_3 is weak isotropic spin) which is determined by physical conditions. This formula essentially describes the embedding of the electromagnetic U(1) in the full gauge group.

Theorem 5, which is our final result, may now be stated in physical language as follows. *An electroweak gauge theory based on a compact connected non-semisimple Lie group admits no stringless magnetic monopoles, except in the case in which the charge operator lies entirely inside the semisimple part of the algebra. The result is independent of the mechanism used to break the symmetry.*

We remark that the exceptional case arises because the result of theorem 5 depends on the assumption that q has a non-zero $\mathscr{K}$ (or hypercharge) component). This is of little physical interest.

8. Conclusion

The non-existence of 't Hooft–Polyakov monopoles in the Weinberg–Salam model—perhaps the most successful of all gauge theories—is a fundamental result in monopole theory. The work described here was prompted by the question of whether it is possible to modify the Weinberg–Salam model in such a way that the non-existence theorem can be circumvented. We have attempted to answer this question by isolating a *minimal* set of assumptions from which the non-existence result can be deduced. The conclusion is that the theorem can be proved purely from the following data concerning the electroweak gauge group G.

(a) G is compact and connected.

(b) G is not semisimple.

(c) The charge operator is not an element of the semisimple part of the algebra of G.

We do *not* need to make any assumptions as to the way in which the symmetry is broken, nor do we require information about the homotopy groups of G or its quotient spaces.

It is very likely that this list can be reduced still further: assumption (a) is almost certainly unnecessary. Thus, the non-existence theorem can be circumvented only by modifying (b) or (c) or both. Modification of (c) is unlikely to be useful: in the Weinberg–Salam case, this would entail a charge formula $q = T_3$, which is clearly unacceptable. We conclude, therefore, that embedding G in some larger, semisimple group (as is done in grand unified theories) is almost certainly the *only* physically acceptable way of circumventing the non-existence theorem. In particular, no modification of the Higgs mechanism can have this effect.

We conclude with a discussion of the relationship of our approach to the more familiar existence theory for monopoles, as discussed for example by Goddard and Olive (1978). The latter is based on the observation that the Higgs fields define mappings from the 2-sphere at infinity to a certain manifold isomorphic to G/H, where G is a gauge group broken by the Higgs mechanism to a subgroup H. This leads to the concept of topological charge, defined in terms of the homotopy group $\pi_2(G/H)$. The standard proof of the non-existence theorem then involves a demonstration that this group is trivial in the Weinberg–Salam case. Although this proof is manifestly independent of the particular way in which the Higgs mechanism is implemented (that is, the actual values of the Higgs fields, their transformation behaviour, and so on are irrelevant), it nevertheless rests on an additional assumption, independent of (a), (b) and (c) above: namely, the assumption that the model contains scalar fields obeying certain specific boundary conditions at spatial infinity. (This is true even if one expresses the topological charge by means of a formula which involves only gauge fields and not Higgs fields, because this formula (Goddard and Olive 1978) is *derived* from the boundary conditions on the covariant derivatives of the Higgs fields.)

If one adds this assumption (together with some additional topological information) to (a), (b) and (c) above, then it becomes possible to relate the two methods of proof. A well known theorem in fibre bundle theory (Kobayashi and Nomizu 1963) essentially states that the H sub-bundles of a given G bundle are determined by cross sections of a certain associated bundle with standard fibre G/H. With respect to a fixed choice of gauge, these cross sections may be interpreted as mappings from the base manifold to G/H. As Madore (1977) pointed out, the Higgs fields may be given a fibre bundle interpretation in this way. Each Higgs field determines a cross section of the bundle with standard fibre G/H, and so fixes a sub-bundle of the given G bundle. Now if the Higgs fields are all homotopically equivalent—that is, if $\pi_2(G/H)$ is zero—it follows that all H sub-bundles of the given G bundle are equivalent. Since every trivial G bundle admits a trivial H sub-bundle, we conclude in this case that all of the sub-bundles are trivial. To summarise, then, we see that if the Higgs fields responsible for the breakdown of G to H are all homotopic, then no trivial G bundle can admit a non-trivial H sub-bundle. This means, as in § 2, that monopoles cannot exist. This discussion establishes the relationship of the present method to the standard one, in the context of the additional assumptions mentioned earlier.

Acknowledgment

The author wishes to take this opportunity to thank Dr B W Wetherilt for several helpful discussions.

References

Actor A 1979 *Rev. Mod. Phys.* **51** 461
Bleecker D 1981 *Gauge Theory and Variational Principles* (Reading, Mass.: Addison-Wesley)
Craigie N S, Goddard P and Nahm W 1982 *Monopoles in Quantum Field Theory* (Singapore: Wold Scientific)
Daniel M and Viallet C M 1980 *Rev. Mod. Phys.* **52** 175
de Rujula A, Georgi H and Glashow S L 1977 *Ann. Phys., NY* **109** 242
Deser S 1976 *Phys. Lett.* **64B** 463
Drechsler W and Mayer M E 1977 *Fiber Bundle Techniques in Gauge Theories* (Berlin: Springer)
Georgi H and Glashow S L 1972 *Phys. Rev. Lett.* **28** 1494
Goddard P, Nuyts J and Olive D 1977 *Nucl. Phys.* B **125** 1
Goddard P and Olive D 1978 *Rep. Prog. Phys.* **41** 1357
Hochschild G P 1965 *The Structure of Lie Groups* (San Fancisco: Holden-Day)
Huang K 1982 *Quarks, Leptons and Gauge Fields* (Singapore: World Sicentific)
Kobayashi S 1972 *Transformation Groups in Differential Geometry* (Berlin: Springer)
Kobayashi S and Nomizu K 1963 *Foundations of Differential Feometry* vol I (New York: Interscience)
—— 1969 *Foundations of Differential Geometry* vol II (New York: Interscience)
Madore J 1977 *Commun. Math. Phys.* **56** 115
Polyakov A M 1974 *JETP Lett.* **20** 194
Quiros M, Mittelbrunn J R and Rodriquez E 1982 *J. Math. Phys.* **23** 873
Ross D K 1983 *J. Math. Phys.* **24** 1814
't Hooft G 1974 *Nucl. Phys.* B **79** 276
Trautman A 1970 *Rep. Math. Phys.* **1** 29
—— 1976 *Rep. Math. Phys.* **10** 297
—— 1979 *Czech. J. Phys.* B **29** 107
Varadarajan, V S 1974 *Lie Groups, Lie Algebras, and Their Representations* (London: Prentice-Hall)
Wu T T and Yang C N 1975 *Phys. Rev.* D **12** 3845